

SCHUR MULTIPLIERS ON $\mathcal{B}(L^p, L^q)$

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ABSTRACT. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces and let $1 \leq p, q \leq +\infty$. We give a definition of Schur multipliers on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ which extends the definition of classical Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. Our main result is a characterization of Schur multipliers in the case $1 \leq q \leq p \leq +\infty$. When $1 < q \leq p < +\infty$, $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if there are a measure space (a probability space when $p \neq q$) (Ω, μ) , $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \langle a(s), b(t) \rangle.$$

Here, $L^\infty(\mu_1, L^r(\mu))$ denotes the Bochner space on Ω_1 valued in $L^r(\mu)$. This result is new, even in the classical case. As a consequence, we give new inclusion relationships between the spaces of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$.

1. INTRODUCTION

If $1 \leq r < +\infty$, we denote by ℓ_r the Banach space of r -summable sequences $(x_i)_{i \geq 1} \subset \mathbb{C}$ (that is, $\sum_i |x_i|^r < +\infty$) endowed with the norm $\|x\|_{\ell_r} = (\sum_i |x_i|^r)^{1/r}$. Let ℓ_∞ be the Banach space of bounded sequences $(y_i)_{i \geq 1} \subset \mathbb{C}$ with the norm $\|y\|_{\ell_\infty} = \sup_i |y_i|$. If $n \in \mathbb{N}$, we denote by ℓ_r^n the n -dimensional versions of the spaces introduced before.

Let $m = (m_{ij})_{i,j \geq 1}$ be a bounded family of complex numbers and let $1 \leq p, q \leq +\infty$. We say that m is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$ if for any matrix $[a_{ij}]_{i,j \geq 1}$ in $\mathcal{B}(\ell_p, \ell_q)$, the matrix $[m_{ij}a_{ij}]_{i,j \geq 1}$ defines an element of $\mathcal{B}(\ell_p, \ell_q)$. An application of the Closed Graph theorem shows that m is a Schur multiplier if and only if the mapping

$$(1) \quad \begin{aligned} T_m : \mathcal{B}(\ell_p, \ell_q) &\longrightarrow \mathcal{B}(\ell_p, \ell_q) \\ [a_{ij}]_{i,j \geq 1} &\longmapsto [m_{ij}a_{ij}]_{i,j \geq 1} \end{aligned}$$

is bounded. By definition, the norm of the Schur multiplier m is the norm of T_m .

There is a well-known characterization of Schur multipliers on $\mathcal{B}(\ell_2)$ (see for instance [11, Theorem 5.1]) which can be extended to the case $\mathcal{B}(\ell_p)$ as follows.

Theorem 1.1. [11, Theorem 5.10] *Let $\phi = (c_{ij})_{i,j \in \mathbb{N}} \subset \mathbb{C}$, $C \geq 0$ be a constant and let $1 \leq p < \infty$. The following are equivalent :*

- (i) ϕ is a Schur multiplier on $\mathcal{B}(\ell_p)$ with norm $\leq C$.
- (ii) There is a measure space (Ω, μ) and elements $(x_j)_{j \in \mathbb{N}}$ in $L^p(\mu)$ and $(y_i)_{i \in \mathbb{N}}$ in $L^{p'}(\mu)$ such that

$$\forall i, j \in \mathbb{N}, c_{ij} = \langle x_j, y_i \rangle \text{ and } \sup_i \|y_i\|_{p'} \sup_j \|x_j\|_p \leq C.$$

Denote by $\mathcal{M}(p, q)$ the space of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$. In [3], Bennett gives some results about the inclusions between the spaces $\mathcal{M}(p, q)$. In the same paper, he also gives a necessary and sufficient condition for a family m to belong to $\mathcal{M}(p, q)$, using the theory of absolutely summing operators. Theorem 1.1 provides a different type of characterization, which is more explicit and useful.

Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ -finite measure spaces. The space $L^2(\Omega_1 \times \Omega_2)$ can be identified with the space $S^2(L^2(\Omega_1), L^2(\Omega_2))$ of Hilbert-Schmidt operators. If $J \in L^2(\Omega_1 \times \Omega_2)$, the operator

$$\begin{aligned} X_J : L^2(\Omega_1) &\longrightarrow L^2(\Omega_2) \\ f &\longmapsto \int_{\Omega_1} J(t, \cdot) f(t) \, d\mu_1(t) \end{aligned}$$

is a Hilbert-Schmidt operator and $\|X_J\|_2 = \|J\|_{L^2}$. Moreover, any element of $S^2(L^2(\Omega_1), L^2(\Omega_2))$ has this form.

Let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. We may associate the operator

$$\begin{aligned} R_\phi : S^2(L^2(\Omega_1), L^2(\Omega_2)) &\longrightarrow S^2(L^2(\Omega_1), L^2(\Omega_2)) \\ X_J &\longmapsto X_{\phi J} \end{aligned}$$

whose norm is equal to $\|\phi\|_\infty$. We say that ϕ is a Schur multiplier on $\mathcal{B}(L^2(\Omega_1), L^2(\Omega_2))$ if R_ϕ extends to a (necessarily unique) bounded operator still denoted by

$$R_\phi : \mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)) \longrightarrow \mathcal{K}(L^2(\Omega_1), L^2(\Omega_2)),$$

where $\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2))$ denotes the space of compact operators from $L^2(\Omega_1)$ into $L^2(\Omega_2)$. When ϕ is a Schur multiplier, the norm of ϕ is by definition the norm of R_ϕ as an operator from $\mathcal{K}(L^2(\Omega_1), L^2(\Omega_2))$ into itself.

A characterization similar to the one in Theorem 1.1 holds in this setting. The following result was established by Peller [9].

Theorem 1.2. *Let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ and $C > 0$. The following are equivalent :*

- (i) ϕ is a Schur multiplier and $\|R_\phi\| < C$.
- (ii) There exist families $(a_i)_{i \geq 1} \subset L^\infty(\Omega_1)$ and $(b_i)_{i \geq 1} \subset L^\infty(\Omega_2)$ such that

$$\operatorname{esssup}_{s \in \Omega_1} \sum_{i=1}^{+\infty} |a_i(s)|^2 < C, \operatorname{esssup}_{t \in \Omega_2} \sum_{i=1}^{+\infty} |b_i(t)|^2 < C$$

and for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \sum_{i=1}^{+\infty} a_i(s) b_i(t).$$

See also [12] for another formulation of this theorem and results about Schur multipliers in the measurable case.

In this article, we define more generally Schur multipliers on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ for some measure spaces (Ω_1, μ_1) and (Ω_2, μ_2) . To any $\phi \in L^\infty(\Omega_1, \Omega_2)$, we associate a linear mapping

$$T_\phi : L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \rightarrow L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$$

and we say that ϕ is a Schur multiplier if T_ϕ is bounded. When $\Omega_1 = \Omega_2 = \mathbb{N}$ with the counting measures, T_ϕ corresponds to (1).

In the case $1 \leq q \leq p \leq +\infty$, we characterize the elements of $L^\infty(\Omega_1 \times \Omega_2)$ which are Schur multipliers on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$. We prove that if $1 < q \leq p < +\infty$, ϕ is a Schur multiplier if and only if there are a measure space (a probability space when $p \neq q$) (Ω, μ) , $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \langle a(s), b(t) \rangle,$$

where $L^\infty(\mu_1, L^r(\mu))$ is the Bochner space valued in $L^r(\mu)$.

This result is new, even in the setting of classical Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$, and is of different nature than the characterization of Bennett. As a consequence, we give in the last section of this article new results of comparisons for the spaces $\mathcal{M}(p, q)$.

1.1. Notations. Let X and Y be Banach spaces.

If $z \in X \otimes Y$, the projective tensor norm of z is defined by

$$\|z\|_\wedge := \inf \left\{ \sum \|x_i\| \|y_i\| \right\},$$

where the infimum runs over all finite families $(x_i)_i$ in X and $(y_i)_i$ in Y such that

$$z = \sum_i x_i \otimes y_i.$$

The completion $X \overset{\wedge}{\otimes} Y$ of $(X \otimes Y, \|\cdot\|_\wedge)$ is called the projective tensor product of X and Y . Note that the projective tensor product is commutative, that is $X \overset{\wedge}{\otimes} Y = Y \overset{\wedge}{\otimes} X$.

The mapping taking any functional $\omega : X \otimes Y \rightarrow \mathbb{C}$ to the operator $u : X \rightarrow Y^*$ defined by $\langle u(x), y \rangle = \omega(x \otimes y)$ for any $x \in X, y \in Y$, induces an isometric identification

$$(2) \quad (X \overset{\wedge}{\otimes} Y)^* = \mathcal{B}(X, Y^*).$$

We refer to [7, Chapter 8, Corollary 2] for this fact.

Let (Ω, μ) be a localizable measure space and let $L^p(\Omega; Y)$ denote the Bochner space of p -integrable functions from Ω into Y . By [7, Chapter 8, Example 10], the natural embedding $L^1(\Omega) \otimes Y \subset L^1(\Omega; Y)$ extends to an isometric isomorphism

$$(3) \quad L^1(\Omega; Y) = L^1(\Omega) \overset{\wedge}{\otimes} Y.$$

By (2), this implies

$$(4) \quad L^1(\Omega; Y)^* = \mathcal{B}(L^1(\Omega), Y^*).$$

Assume that Y^* has the Radon-Nikodym property (in short, Y^* has RNP). In this case,

$$L^1(\Omega, Y)^* = L^\infty(\Omega, Y^*).$$

The latter implies that

$$(5) \quad L^\infty(\Omega, Y^*) = \mathcal{B}(L^1(\Omega), Y^*),$$

and the isometric isomorphism is given by

$$\begin{aligned} L^\infty(\Omega, Y^*) &\longrightarrow \mathcal{B}(L^1(\Omega), Y^*). \\ g &\longmapsto \left[f \in L^1(\Omega) \mapsto \int_{\Omega} f(t)g(t)d\mu(t) \right] \end{aligned}$$

Assume now that $Y = L^1(\Omega')$ where (Ω', μ') is a localizable measure space. Then, an application of Fubini Theorem gives

$$L^1(\Omega, L^1(\Omega')) = L^1(\Omega \times \Omega').$$

Using equality (3), we deduce that

$$(6) \quad \mathcal{B}(L^1(\Omega), L^\infty(\Omega')) = L^\infty(\Omega \times \Omega'),$$

and the correspondence is given by

$$\begin{aligned} L^\infty(\Omega \times \Omega') &\longrightarrow \mathcal{B}(L^1(\Omega), L^\infty(\Omega')). \\ \psi &\longmapsto \left[f \in L^1(\Omega) \mapsto \int_{\Omega} f(t)\psi(t, \cdot)d\mu(t) \right] \end{aligned}$$

For $\psi \in L^\infty(\Omega \times \Omega')$, denote by u_ψ the corresponding element of $\mathcal{B}(L^1(\Omega), L^\infty(\Omega'))$.

If $z = \sum_i x_i \otimes y_i \in X \otimes Y$, $x^* \in X^*$ and $y^* \in Y^*$, we write

$$\langle z, x^* \otimes y^* \rangle = \sum_i x^*(x_i)y^*(y_i).$$

Then, the injective tensor norm of $z \in X \otimes Y$ is given by

$$\|z\|_\vee = \sup_{\|x^*\| \leq 1, \|y^*\| \leq 1} |\langle z, x^* \otimes y^* \rangle|.$$

The completion $X \overset{\vee}{\otimes} Y$ of $(X \otimes Y, \|\cdot\|_\vee)$ is called the injective tensor product of X and Y .

In this paper, we will often identify $X^* \otimes Y$ with the finite rank operators from X into Y as follow. If $u = \sum_i x_i^* \otimes y_i \in X^* \otimes Y$, we define $\tilde{u} : X \rightarrow Y$ by

$$(7) \quad \tilde{u}(x) = \sum_i x_i^*(x)y_i, \forall x \in X.$$

Then, it is easy to check that $\|u\|_\vee = \|\tilde{u}\|_{\mathcal{B}(X, Y)}$.

Moreover, if Y has the approximation property (see e.g. [6] for the definition), [6, Theorem 1.4.21] gives the isometric identification

$$X^* \overset{\vee}{\otimes} Y = \mathcal{K}(X, Y)$$

where $\mathcal{K}(X, Y)$ denotes the space of compact operators from X into Y .

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two localizable measure spaces. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then $L^q(\Omega_2)$ has the approximation property so that we have

$$(8) \quad L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) = \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)).$$

Finally, if we assume that $1 < p, q < +\infty$, then by [5, Theorem 2.5] and (2),

$$(9) \quad (L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2))^{**} = (L^p(\Omega_1) \overset{\wedge}{\otimes} L^{q'}(\Omega_2))^* = \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2)).$$

2. DEFINITION OF SCHUR MULTIPLIERS ON $\mathcal{B}(L^p, L^q)$

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two localizable measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leq p, q \leq \infty$ and denote by p' and q' their conjugate exponents. Let

$$T_\phi : L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \rightarrow \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

be defined for any elementary tensor $f \otimes g \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$ by

$$[T_\phi(f \otimes g)](h) = \left(\int_{\Omega_1} \phi(s, \cdot) f(s) h(s) d\mu_1(s) \right) g(\cdot) \in L^q(\Omega_2),$$

for all $h \in L^p(\Omega_1)$.

We have an inclusion

$$L^{p'}(\Omega_1) \otimes L^q(\Omega_2) \subset L^{p'}(\Omega_1, L^q(\Omega_2))$$

given by $f \otimes g \mapsto [s \in \Omega_1 \mapsto f(s)g]$. Under this identification, T_ϕ is the multiplication by ϕ . Note that $L^{p'}(\Omega_1, L^q(\Omega_2))$ is invariant by multiplication by an element of $L^\infty(\Omega_1 \times \Omega_2)$ and that we have a contractive inclusion

$$L^{p'}(\Omega_1, L^q(\Omega_2)) \subset L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2).$$

Therefore, T_ϕ is valued in $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$. Using the identification

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \subset \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

given by (7), we deduce that the elements of $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$ are compact operators as limits of finite rank operators for the operator norm.

Definition 2.1. We say that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if there exists a constant $C \geq 0$ such that for all $u \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2)$,

$$\|T_\phi(u)\|_{\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))} \leq \|u\|_\vee,$$

that is, if T_ϕ extends to a bounded operator

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \rightarrow L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2).$$

In this case, the norm of ϕ is by definition the norm of T_ϕ .

Remark 2.2. By \mathcal{E}_1 (resp. \mathcal{E}_2) we denote the space of simple functions on Ω_1 (resp. Ω_2). By density of $\mathcal{E}_1 \otimes \mathcal{E}_2$ in $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$, T_ϕ extends to a bounded operator from $L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$ into itself if and only if it is bounded on $\mathcal{E}_1 \otimes \mathcal{E}_2$ equipped with the injective tensor norm.

Assume that $1 < p, q < +\infty$. By (8) we have

$$L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) = \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)),$$

so that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if T_ϕ extends to a bounded operator

$$T_\phi : \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)) \rightarrow \mathcal{K}(L^p(\Omega_1), L^q(\Omega_2)).$$

In this case, considering the bi-adjoint of T_ϕ , we obtain by (9) a w^* -continuous mapping

$$\tilde{T}_\phi : \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2)) \rightarrow \mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$$

which extends T_ϕ . This explains the terminology ' ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ '.

Classical Schur multipliers : Assume that $\Omega_1 = \Omega_2 = \mathbb{N}$ and that μ_1 and μ_2 are the counting measures. An element $\phi \in L^\infty(\mathbb{N}^2)$ is given by a family $c = (c_{ij})_{i,j \in \mathbb{N}}$ of complex numbers, where $c_{ij} = \phi(j, i)$. In this situation, the mapping T_ϕ is nothing but the classical Schur multiplier

$$A = [a_{ij}]_{i,j \geq 1} \in \mathcal{B}(\ell_p, \ell_q) \mapsto [c_{ij}a_{ij}]_{i,j \geq 1}.$$

When this mapping is bounded from $\mathcal{B}(\ell_p, \ell_q)$ into itself, we will denote it by T_c .

Notations : If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $n \in \mathbb{N}^*$, we denote by $\mathcal{A}_{n,\Omega}$ the collection of n -tuples (A_1, \dots, A_n) of pairwise disjoint elements of \mathcal{F} such that

$$\text{for all } 1 \leq i \leq n, 0 < \mu(A_i) < +\infty.$$

If $A = (A_1, \dots, A_n) \in \mathcal{A}_{n,\Omega}$ and $1 \leq p \leq +\infty$, denote by $S_{A,p}$ the subspace of $L^p(\Omega)$ generated by $\chi_{A_1}, \dots, \chi_{A_n}$. Then $S_{A,p}$ is 1-complemented in $L^p(\Omega)$, and a norm one projection from $L^p(\Omega)$ into $S_{A,p}$ is given by the conditional expectation

$$(10) \quad \begin{aligned} P_{A,p} : L^p(\Omega) &\longrightarrow L^p(\Omega). \\ f &\longmapsto \sum_{i=1}^n \frac{1}{\mu(A_i)} \left(\int_{A_i} f \right) \chi_{A_i} \end{aligned}$$

Note that the mapping

$$(11) \quad \begin{aligned} \varphi_{A,p} : S_{A,p} &\longrightarrow \ell_p^n. \\ f = \sum_i a_i \chi_{A_i} &\longmapsto (a_i (\mu_1(A_i))^{1/p})_{i=1}^n \end{aligned}$$

is an isometric isomorphism between $S_{A,p}$ and ℓ_p^n .

Proposition 2.3. *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. The following are equivalent :*

- (i) ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$.

(ii) For all $n, m \in \mathbb{N}^*$, for all $A = (A_1, \dots, A_n) \in \mathcal{A}_{n, \Omega_1}, B = (B_1, \dots, B_m) \in \mathcal{A}_{m, \Omega_2}$, write

$$\phi_{ij} = \frac{1}{\mu_1(A_j)\mu_2(B_i)} \int_{A_j \times B_i} \phi \, d\mu_1 d\mu_2.$$

Then the Schur multipliers on $\mathcal{B}(\ell_p^n, \ell_q^m)$ associated with the families $\phi_{A,B} = (\phi_{ij})$ are uniformly bounded with respect to n, m, A and B .

In this case, $\|T_\phi\| = \sup_{n,m,A,B} \|T_{\phi_{A,B}}\| < +\infty$.

Proof. (i) \Rightarrow (ii). Assume first that ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ with $\|T_\phi\| \leq 1$. Let $n, m \in \mathbb{N}^*, A = (A_1, \dots, A_n) \in \mathcal{A}_{n, \Omega_1}$ and $B = (B_1, \dots, B_m) \in \mathcal{A}_{m, \Omega_2}$. Let $c = \sum_{i,j} c(i,j) e_j \otimes e_i \in \ell_p^n \otimes \ell_q^m \simeq \mathcal{B}(\ell_p^n, \ell_q^m)$.

Let $\varphi_{A,p} : S_{A,p} \rightarrow \ell_p^n$ and $\psi_{B,q} : S_{B,q} \rightarrow \ell_q^m$ be the isometries defined in (11). Then $\tilde{c} := \psi_{B,q}^{-1} \circ c \circ \varphi_{A,p} : S_{A,p} \rightarrow S_{B,q}$ satisfies $\|\tilde{c}\| = \|c\|$ and we have

$$\begin{aligned} \tilde{c} &= \sum_{i,j} \frac{c(i,j)}{\mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}} \chi_{A_j} \otimes \chi_{B_i} \\ &:= \sum_{i,j} \tilde{c}(i,j) \chi_{A_j} \otimes \chi_{B_i}, \end{aligned}$$

where $\tilde{c}(i,j) = \frac{c(i,j)}{\mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}}$.

The operator $u := \psi_{B,q} \circ P_{B,q} \circ T_\phi(\tilde{c})|_{S_{A,p}} \circ \varphi_{A,p}^{-1} : \ell_p^n \rightarrow \ell_q^m$ satisfies

$$\|u\| \leq \|T_\phi(\tilde{c})\|$$

and by assumption

$$\|T_\phi(\tilde{c})\| \leq \|\tilde{c}\|$$

so that

$$(12) \quad \|u\| \leq \|\tilde{c}\| = \|c\|.$$

Let us prove that $u = T_{\phi_{A,B}}(c)$ where $T_{\phi_{A,B}}$ is the Schur multiplier associated with the family (ϕ_{ij}) .

Write $u(i,j) := \psi_{B,q} \circ P_{B,q} \circ T_\phi(\chi_{A_j} \otimes \chi_{B_i})|_{S_{A,p}} \circ \varphi_{A,p}^{-1}$. We have

$$u = \sum_{i,j} \tilde{c}(i,j) u(i,j).$$

Let $1 \leq k \leq n$.

$$\begin{aligned} [u(i,j)](e_k) &= [\psi_{B,q} \circ P_{B,q} \circ T_\phi(\chi_{A_j} \otimes \chi_{B_i})|_{S_{A,p}}] \left(\frac{1}{\mu_1(A_k)^{1/p}} \chi_{A_k} \right) \\ &= \frac{1}{\mu_1(A_k)^{1/p}} [\psi_{B,q} \circ P_{B,q}] \left(\chi_{B_i}(\cdot) \int_{\Omega_1} \phi(s, \cdot) \chi_{A_j}(s) \chi_{A_k}(s) d\mu_1(s) \right) \end{aligned}$$

so that $[u(i, j)](e_k) = 0$ if $k \neq j$ and if $k = j$ then

$$\begin{aligned} [u(i, j)](e_k) &= \frac{1}{\mu_1(A_k)^{1/p}} [\psi_{B,q} \circ P_{B,q}] \left(\chi_{B_i}(\cdot) \int_{A_j} \phi(s, \cdot) d\mu_1(s) \right) \\ &= \frac{1}{\mu_1(A_k)^{1/p} \mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) \psi_q(\chi_{B_i}) \\ &= \frac{1}{\mu_1(A_k)^{1/p} \mu_2(B_i)^{1/q'}} \left(\int_{A_j \times B_i} \phi \right) e_i \end{aligned}$$

It follows that

$$\begin{aligned} u &= \sum_{i,j} \frac{c(i, j)}{\mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}} \frac{1}{\mu_1(A_j)^{1/p} \mu_2(B_i)^{1/q'}} \left(\int_{A_j \times B_i} \phi \right) e_j \otimes e_i \\ &= \sum_{i,j} \frac{c(i, j)}{\mu_1(A_j) \mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) e_j \otimes e_i \\ &= \sum_{i,j} \phi_{ij} c(i, j) e_j \otimes e_i \end{aligned}$$

that is, $u = T_{\phi_{A,B}}(c)$. We conclude thanks to the inequality (12).

(ii) \Rightarrow (i). Assume now that the assertion (ii) is satisfied and show that ϕ is a Schur multiplier. By Remark 2.2, we just need to show that T_ϕ is bounded on $\mathcal{E}_1 \otimes \mathcal{E}_2$. Let $v \in \mathcal{E}_1 \otimes \mathcal{E}_2$ and write $\alpha = \sup_{n,m,A,B} \|T_c\|$. We will show that $\|T_\phi(v)\| \leq \alpha \|v\|$. By density, it is enough to prove that for any $h_1 \in \mathcal{E}_1, h_2 \in \mathcal{E}_2$,

$$(13) \quad |\langle [T_\phi(v)](h_1), h_2 \rangle_{L^q, L^{q'}}| \leq \alpha \|v\|_{\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))} \|h_1\|_{L^p(\Omega_1)} \|h_2\|_{L^{q'}(\Omega_2)}.$$

By assumption, there exist $n, m \in \mathbb{N}^*, A = (A_1, \dots, A_n) \in \mathcal{A}_{n, \Omega_1}, B = (B_1, \dots, B_m) \in \mathcal{A}_{m, \Omega_2}$ and complex numbers $v(i, j), a_i, b_j$ such that

$$v = \sum_{i,j} v(i, j) \chi_{A_j} \otimes \chi_{B_i}, h_1 = \sum_j a_j \chi_{A_j} \text{ and } h_2 = \sum_i b_i \chi_{B_i}.$$

Equation (13) can be rewritten as

$$(14) \quad \left| \sum_{i,j} v(i, j) a_j b_i \left(\int_{A_j \times B_i} \phi \right) \right| \leq \alpha \|v\| \|h_1\|_{L^p(\Omega_1)} \|h_2\|_{L^{q'}(\Omega_2)}.$$

Consider $\tilde{v} := \psi_{B,q} \circ v \circ \varphi_{A,p}^{-1} : \ell_p^n \rightarrow \ell_q^m$ and $z := \psi_{B,q} \circ P_{B,q} \circ T_\phi(v)|_{S_{A,p}} \circ \phi_{A,p}^{-1} : \ell_p^n \rightarrow \ell_q^m$. The computations made in the first part of the proof show that $z = T_m(\tilde{v})$ where m is the family (ϕ_{ij}) .

Now, let $x := \varphi_{A,p}(h_1)$ and $y := \psi_{B,q'}(h_2)$. Since T_m is bounded with norm smaller than α we have

$$(15) \quad |\langle [T_m(\tilde{v})](x), y \rangle_{\ell_q^m, \ell_{q'}^m}| \leq \alpha \|\tilde{c}\|_{\mathcal{B}(\ell_p^n, \ell_q^m)} \|x\|_{\ell_p^n} \|y\|_{\ell_{q'}^m}.$$

An easy computation shows that the left-hand side on this equality is nothing but the left-hand side of the inequality (14). Finally, the right-hand side of the inequalities (14) and (15) are equal, which concludes the proof. \square

3. (p, q) -FACTORABLE OPERATORS

Let X and Y be Banach spaces.

3.1. Dual norm. [4, Chapter 15]. Let $M \subset X$ and $N \subset Y$ be finite dimensional subspaces (in short, f.d.s). If $u = \sum_{i=1}^n x_i \otimes y_i \in M \otimes N$ and $v = \sum_{j=1}^m x_j^* \otimes y_j^* \in M^* \otimes N^*$ we set

$$\langle v, u \rangle = \sum_{i,j} \langle x_j^*, x_i \rangle \langle y_j^*, y_i \rangle.$$

Let α be a tensor norm on tensor products of finite dimensional spaces. We define, for $z \in M \otimes N$,

$$\alpha'(z, M, N) = \sup \{ |\langle v, u \rangle| \mid v \in M^* \otimes N^*, \alpha(v) \leq 1 \}.$$

Now, for $z \in X \otimes Y$, we set

$$\alpha'(z, X, Y) = \inf \{ \alpha'(z, M, N) \mid M \subset X, N \subset Y \text{ f.d.s., } z \in M \otimes N \}.$$

α' defines a tensor norm on $X \otimes Y$, called the dual norm of α .

In the sequel, we will write $\alpha'(z)$ instead of $\alpha'(z, X, Y)$ for the norm of an element $z \in X \otimes Y$ when there is no possible confusion.

3.2. Lapresté norms. [4, Proposition 12.5]. Let $s \in [1, \infty]$. If $x_1, x_2, \dots, x_n \in X$, we define

$$w_s(x_i, X) := \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^n |\langle x^*, x_i \rangle|^s \right)^{1/s}.$$

Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \geq 1$ and take $r \in [1, \infty]$ such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Denote by p' and q' the conjugate of p and q . For $z \in X \otimes Y$, we define

$$\alpha_{p,q}(z) = \inf \left\{ \|(\lambda_i)_i\|_{\ell_r} w_{q'}(x_i, X) w_{p'}(y_i, Y) \mid z = \sum_{i=1}^n \lambda_i x_i \otimes y_i \right\}.$$

Then $\alpha_{p,q}$ is a norm on $X \otimes Y$ and we denote by $X \otimes_{\alpha_{p,q}} Y$ its completion.

3.3. (p, q) –Factorable operators. If $T \in \mathcal{B}(X, Y^*)$ and $\xi = \sum_i x_i \otimes y_i \in X \otimes Y$, then in accordance with (2) we set

$$\langle T, \xi \rangle = \sum_i \langle T(x_i), y_i \rangle.$$

Definition 3.1. Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Let $T \in \mathcal{B}(X, Y^*)$. We say that $T \in \mathcal{L}_{p,q}(X, Y^*)$ if there exists a constant $C \geq 0$ such that

$$(16) \quad \forall \xi \in X \otimes Y, \quad |\langle T, \xi \rangle| \leq C \alpha'_{p,q}(\xi).$$

In this case, we write $L_{p,q}(T) = \inf \{C \mid C \text{ satisfying (16)}\}$.

Then $(\mathcal{L}_{p,q}(X, Y^*), L_{p,q})$ is a Banach space, called the space of (p, q) –Factorable operators.

For a general definition of the spaces $\mathcal{L}_{p,q}(X, Y)$ (including the case when the range is not a dual space), see [4, Chapter 17].

Since Y^* is 1-complemented in its bidual, [4, Theorem 18.11] gives the following result.

Theorem 3.2. Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Let $T \in \mathcal{B}(X, Y^*)$. The two following statements are equivalent :

(i) $T \in \mathcal{L}_{p,q}(X, Y^*)$.

(ii) There are a measure space (Ω, μ) (a probability space when $\frac{1}{p} + \frac{1}{q} > 1$), operators $R \in \mathcal{B}(X, L^{q'}(\mu))$ and $S \in \mathcal{B}(L^p(\mu), Y^*)$ such that $T = S \circ I \circ R$

$$\begin{array}{ccc} X & \xrightarrow{T} & Y^* \\ R \downarrow & & \uparrow S \\ L^{q'}(\mu) & \xrightarrow{I} & L^p(\mu) \end{array}$$

where $I : L^{q'}(\mu) \rightarrow L^p(\mu)$ is the inclusion mapping (well defined because $q' \geq p$).

In this case, $L_{p,q}(T) = \inf \|S\| \|R\|$ over all such factorizations.

Remark 3.3. Here we consider the case when $\frac{1}{p} + \frac{1}{q} = 1$. Denote by p' the conjugate exponent of p . We have $T \in \mathcal{L}_{p,p'}(X, Y^*)$ if and only if there are a measure space (Ω, μ) , operators $R \in \mathcal{B}(X, L^p(\mu))$ and $S \in \mathcal{B}(L^p(\mu), Y^*)$ such that $T = SR$

$$\begin{array}{ccc} X & \xrightarrow{T} & Y^* \\ R \searrow & & \nearrow S \\ & L^p(\mu) & \end{array}$$

We usually write $\Gamma_p(X, Y^*)$ instead of $\mathcal{L}_{p,p'}(X, Y^*)$. Such operators are called p –factorable.

Remark 3.4. Suppose that $X = L^1(\lambda)$ and $Y = L^1(\nu)$ for some localizable measure spaces (Ω_1, λ) and (Ω_2, ν) . Consider $T \in \mathcal{B}(L^1(\lambda), L^\infty(\nu))$. By (6), there exists $\psi \in L^\infty(\lambda \times \nu)$ such that

$$T = u_\psi.$$

(See subsection 1.1 for the notation.)

- (i) If $1 < q < +\infty$, $L^{q'}(\mu)$ has RNP so by (5),

$$\mathcal{B}(L^1(\lambda), L^{q'}(\mu)) = L^\infty(\lambda, L^{q'}(\mu)).$$

It means that if $R \in \mathcal{B}(X, L^{q'}(\mu))$, there exists $a \in L^\infty(\lambda, L^{q'}(\mu))$ such that

$$\forall f \in L^1(\lambda), R(f) = \int_{\Omega_1} f(s)a(s)d\lambda(s).$$

- (ii) If $1 < p < +\infty$, then using (2), (3) and (4) we obtain

$$B(L^p(\mu), L^\infty(\nu)) = (L^p(\mu) \hat{\otimes} L^1(\nu))^* = L^\infty(\nu, L^{p'}(\mu)).$$

Thus, if $S \in \mathcal{B}(L^p(\mu), L^\infty(\nu))$, there exists $b \in L^\infty(\nu, L^{p'}(\mu))$ such that

$$\forall g \in L^p(\lambda), S(g)(\cdot) = \langle g, b(\cdot) \rangle.$$

We deduce that if $1 < p, q < +\infty$, there exist $a \in L^\infty(\lambda, L^{q'}(\mu))$ and $b \in L^\infty(\nu, L^{p'}(\mu))$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\psi(s, t) = \langle a(s), b(t) \rangle.$$

If T satisfies Theorem 3.2, the latter implies that for all $f \in L^1(\lambda)$,

$$T(f) = \int_{\Omega_1} \langle a(s), b(\cdot) \rangle f(s) ds.$$

Using the same identifications we have for the following cases :

- (1) If $q = 1$ and $1 < p < +\infty$, then there exist $a \in L^\infty(\lambda \times \mu)$ and $b \in L^\infty(\nu, L^{p'}(\mu))$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\psi(s, t) = \langle a(s, \cdot), b(t) \rangle.$$

- (2) If $1 < q < +\infty$ and $p = +\infty$, then there exist $a \in L^\infty(\lambda, L^{q'}(\mu))$ and $b \in L^\infty(\nu \times \mu)$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\psi(s, t) = \langle a(s), b(t, \cdot) \rangle.$$

- (3) If $q = 1$ and $p = +\infty$, then there exist $a \in L^\infty(\lambda \times \mu)$ and $b \in L^\infty(\nu \times \mu)$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\psi(s, t) = \langle a(s, \cdot), b(t, \cdot) \rangle.$$

3.4. Finite dimensional case. If X and Y are finite dimensional, it follows from the very definition of the dual norm that

$$X \otimes_{\alpha'_{p,q}} Y = (X^* \otimes_{\alpha_{p,q}} Y^*)^*.$$

The next theorem describes the elements of this space.

Theorem 3.5. [4, Theorem 19.2] *Let E and F be Banach spaces. Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \geq 1$ and $K \subset B_{E^*}$ and $L \subset B_{F^*}$ weak- $*$ -compact norming sets for E and F , respectively.*

For $\phi : E \otimes F \rightarrow \mathbb{C}$ the following two statements are equivalent:

(i) $\phi \in (E \otimes_{\alpha_{p,q}} F)^*$.

(ii) *There are a constant $A \geq 0$ and normalized Borel-Radon measures μ on K and ν on L such that for all $x \in E$ and $y \in F$,*

$$(17) \quad |\langle \phi, x \otimes y \rangle| \leq A \left(\int_K |\langle x^*, x \rangle|^{q'} d\mu(x^*) \right)^{1/q'} \left(\int_L |\langle y^*, y \rangle|^{p'} d\nu(y^*) \right)^{1/p'}$$

(if the exponent is ∞ , we replace the integral by the norm).

In this case, $\|\phi\|_{(E \otimes_{\alpha_{p,q}} F)^*} = \inf \{A \mid A \text{ as in (ii)}\}$.

This theorem will allow us to describe the predual of $\mathcal{L}_{p,q}(\ell_1^n, \ell_\infty^m)$, $n, m \in \mathbb{N}$. Let us apply the previous theorem with $E = \ell_\infty^n$ and $F = \ell_\infty^m$. Take $T \in \ell_1^n \otimes_{\alpha'_{p,q}} \ell_1^m = (\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*$ and let

$$T = \sum_{i=1}^n \sum_{j=1}^m T(i, j) e_i \otimes e_j$$

be a representation of T . In the previous theorem, we can take $K = \{1, 2, \dots, n\}$ and $L = \{1, 2, \dots, m\}$. In this case, a normalized Borel-Radon measure μ on K is nothing but a sequence $\mu = (\mu_1, \dots, \mu_n)$ where, for all i , $\mu_i := \mu(\{i\}) \geq 0$ and $\sum_i \mu_i = 1$. Similarly, $\nu = (\nu_1, \dots, \nu_m)$ where, for all i , $\nu_i \geq 0$ and $\sum_i \nu_i = 1$. In this case, the inequality (17) means that for all sequences of complex numbers $x = (x_i)_{i=1}^n, y = (y_j)_{j=1}^m$,

$$\left| \sum_{i=1}^n \sum_{j=1}^m T(i, j) x_i y_j \right| \leq A \left(\sum_{k=1}^n |x_k|^{q'} \mu_k \right)^{1/q'} \left(\sum_{k=1}^m |y_k|^{p'} \nu_k \right)^{1/p'}.$$

Set $\alpha_k = x_k \mu_k^{1/q'}$, $\beta_k = y_k \nu_k^{1/p'}$ and define, for $1 \leq i \leq n, 1 \leq j \leq m$, $c(i, j)$ such that $T(i, j) = c(i, j) \mu_i^{1/q'} \nu_j^{1/p'}$ (we can assume $\mu_i > 0$ and $\nu_j > 0$). Then, the previous inequality becomes

$$\left| \sum_{i=1}^n \sum_{j=1}^m c(i, j) \beta_j \alpha_i \right| \leq A \|\alpha\|_{\ell_{q'}^n} \|\beta\|_{\ell_{p'}^m}.$$

This means that the operator $c : \ell_{q'}^n \rightarrow \ell_p^m$ whose matrix is $[c(i, j)]_{1 \leq j \leq m, 1 \leq i \leq n}$ has a norm smaller than A . Moreover, if we see T as a mapping from ℓ_∞^n into ℓ_1^m the relation between T and c means that T admits the following factorization

$$\begin{array}{ccc}
\ell_\infty^n & \xrightarrow{T} & \ell_1^m \\
d_\mu \downarrow & & \uparrow d_\nu \\
\ell_{q'}^n & \xrightarrow{c} & \ell_p^m
\end{array}$$

where d_μ and d_ν are the operators of multiplication by $\mu = (\mu_1^{1/q'}, \dots, \mu_n^{1/q'})$ and $\nu = (\nu_1^{1/p'}, \dots, \nu_m^{1/p'})$. Those operators have norm 1.

Therefore, it is easy to check that

$$(18) \quad \|T\|_{(\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*} = \inf \{ \|c\| \mid T = d_\nu \circ c \circ d_\mu \}.$$

The elements of $(\ell_\infty^n \otimes_{\alpha_{p,q}} \ell_\infty^m)^*$ are called (q', p') -dominated operators. For more informations about this space in the infinite dimensional case (it is the predual of $\mathcal{L}_{p,q}$), see for instance [4, Chapter 19].

By (18) and the fact that $\mathcal{L}_{p,q}(\ell_1^n, \ell_\infty^n) = (\ell_1^n \otimes_{\alpha'_{p,q}} \ell_1^m)^*$, we get the following result.

Proposition 3.6. *Let $v = [v_{ij}] : \ell_1^n \rightarrow \ell_\infty^m$. Then*

$$L_{p,q}(v) = \sup |Tr(vu)|$$

where the supremum runs over all $u : \ell_\infty^m \rightarrow \ell_1^n$ admitting the factorization

$$\begin{array}{ccc}
\ell_\infty^m & \xrightarrow{u} & \ell_1^n \\
d_\mu \downarrow & & \uparrow d_\nu \\
\ell_{p'}^m & \xrightarrow{c} & \ell_q^n
\end{array}$$

with $\|d_\mu\| \leq 1$, $\|d_\nu\| \leq 1$ and $\|c\| \leq 1$.

Equivalently,

$$L_{p,q}(v) = \sup \left\{ \left| \sum_{i=1}^m \sum_{j=1}^n v_{ij} c_{ji} \mu_i \nu_j \right| \mid \|c : \ell_{p'}^m \rightarrow \ell_q^n\| \leq 1, \|\mu\|_{\ell_{p'}^m} \leq 1, \|\nu\|_{\ell_q^n} \leq 1 \right\}.$$

4. THE MAIN RESULT

4.1. Schur multipliers and factorization. Let p, q be two positive numbers such that $1 \leq q \leq p \leq \infty$. This condition is equivalent to $p, q \in [1, \infty]$ with $\frac{1}{q} + \frac{1}{p'} \geq 1$, so that we can consider the space $\mathcal{L}_{q,p'}$.

The following results will allow us to give a description of the functions ϕ which are Schur multipliers.

Lemma 4.1. *Let X, Y be Banach spaces and let $E \subset X, F \subset Y$ be 1-complemented subspaces of X and Y . For any $v \in E \otimes F$, denote by $\tilde{\alpha}'_{q,p'}(v)$ the $\alpha'_{q,p'}$ -norm of v as an element of $E \otimes F$ and by $\alpha'_{q,p'}(v)$ the $\alpha'_{q,p'}$ -norm of v as an element of $X \otimes Y$. Then*

$$\tilde{\alpha}'_{q,p'}(v) = \alpha'_{q,p'}(v).$$

Proof. The inequality $\tilde{\alpha}'_{q,p'}(v) \geq \alpha'_{q,p'}(v)$ is easy to prove. For the converse inequality, take $v = \sum_k e_k \otimes f_k \in E \otimes F$ such that $\alpha'_{q,p'}(v) < 1$ and show that $\tilde{\alpha}'_{q,p'}(v) < 1$. By assumption, there exists $M \subset X$ and $N \subset Y$ finite dimensional subspaces such that $v \in M \otimes N$ and

$$\alpha'(v, M, N) < 1.$$

By assumption, there exist two norm one projections P and Q respectively from X onto E and from Y onto F . Set $M_1 = P(M) \subset E$ and $N_1 = Q(N) \subset F$. M_1 and N_1 are finite dimensional. Moreover, since $v \in E \otimes F$, it is easy to check that $(P \otimes Q)(v) = v$, where, for all $c = \sum_l a_l \otimes b_l \in X \otimes Y$,

$$(P \otimes Q)(c) = \sum_l P(a_l) \otimes Q(b_l).$$

Thus, $v \in M_1 \otimes N_1$. We will show that $\alpha'_{q,p'}(v, M_1, N_1) < 1$.

Let $z = \sum_{j=1}^m x_j^* \otimes y_j^* \in M_1^* \otimes N_1^*$ be such that $\alpha_{q,p'}(z) < 1$ and show that $|\langle v, z \rangle| \leq \alpha'_{q,p'}(v)$, so that $\alpha'_{q,p'}(v, M_1, N_1) \leq 1$.

Let $1 \leq r \leq \infty$ such that

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{p'} - 1.$$

The condition $\alpha_{q,p'}(z) < 1$ in $M_1^* \otimes N_1^*$ implies that z admits a representation $z = \sum_{j=1}^m \lambda_j m_j^* \otimes n_j^*$ where $m_j^* \in M_1^*, n_j^* \in N_1^*$ and

$$\|(\lambda_j)_j\|_{\ell_r} w_p(m_j^*, M_1^*) w_{q'}(n_j^*, N_1^*) < 1.$$

Set $\tilde{z} := \sum_{j=1}^m \lambda_j P^*(m_j^*) \otimes Q^*(n_j^*)$ in $M^* \otimes N^*$. It is easy to check that

$$w_p(P^*(m_j^*), M^*) \leq w_p(m_j^*, M_1^*) \quad \text{and} \quad w_{q'}(Q^*(n_j^*), N^*) \leq w_{q'}(n_j^*, N_1^*).$$

Therefore, $\alpha_{q,p'}(\tilde{z}, M^*, N^*) < 1$. Then, the condition $\alpha'_{q,p'}(v, M, N) < 1$ implies that

$$|\langle v, \tilde{z} \rangle| \leq \alpha'_{q,p'}(v).$$

Finally, we have

$$\begin{aligned} \langle v, \tilde{z} \rangle &= \sum_{j,k} \lambda_j \langle P^*(m_j^*), e_k \rangle \langle Q^*(n_j^*), f_k \rangle \\ &= \sum_{j,k} \lambda_j \langle m_j^*, P(e_k) \rangle \langle n_j^*, Q(f_k) \rangle \\ &= \sum_{j,k} \lambda_j \langle m_j^*, e_k \rangle \langle n_j^*, f_k \rangle = \langle v, z \rangle, \end{aligned}$$

and therefore

$$|\langle v, z \rangle| \leq \alpha'_{q,p'}(v).$$

This proves that $\tilde{\alpha}'_{q,p'}(v) < 1$. □

We recall that if $\phi \in L^\infty(\Omega_1 \times \Omega_2)$, we denote by u_ϕ the mapping

$$\begin{aligned} u_\phi : L^1(\Omega_1) &\longrightarrow L^\infty(\Omega_2). \\ f &\longmapsto \int_{\Omega_1} \phi(s, \cdot) f(s) \, d\mu_1(s) \end{aligned}$$

Theorem 4.2. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two localizable measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leq q \leq p \leq \infty$. Then ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$ if and only if the operator u_ϕ belongs to $\mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$. Moreover,*

$$\|T_\phi\| = L_{q,p'}(u_\phi).$$

Proof. Assume first that T_ϕ extends to a bounded operator

$$T_\phi : L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2) \rightarrow L^{p'}(\Omega_1) \overset{\vee}{\otimes} L^q(\Omega_2)$$

with norm ≤ 1 . To prove that $u_\phi \in \mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$ with $L_{q,p'}(u_\phi) \leq 1$, we have to show that for any $v = \sum_k f_k \otimes g_k \in L^1(\Omega_1) \otimes L^1(\Omega_2)$ with $\alpha'_{q,p'}(v) < 1$ we have

$$|u_\phi(v)| = \left| \sum_k \langle u_\phi(f_k), g_k \rangle \right| \leq 1.$$

By density, we can assume that f_k, g_k are simple functions. Hence, with the notations introduced in Section 2 there exist $n, m \in \mathbb{N}^*$, $A = (A_1, \dots, A_n) \in \mathcal{A}_{n, \Omega_1}$ and $B = (B_1, \dots, B_m) \in \mathcal{A}_{m, \Omega_2}$ such that, for all k , $f_k \in S_{A,1}$ and $g_k \in S_{B,1}$.

By Lemma 4.1, the $\alpha'_{q,p}$ -norm of v as an element of $S_{A,1} \otimes S_{B,1}$ is less than 1.

Let $\varphi_{A,1} : S_{A,1} \rightarrow \ell_1^n$ and $\psi_{B,1} : S_{B,1} \rightarrow \ell_1^m$ the isomorphisms defined in (11). Set $v' = \sum_k \varphi_{A,1}(f_k) \otimes \psi_{B,1}(g_k) \in \ell_1^n \otimes \ell_1^m$. Since $\varphi_{A,1}$ and $\psi_{B,1}$ are isometries, we have $\alpha'_{q,p'}(v') < 1$. Using the identification (7), we obtain by (18) that v' admits a factorization

$$\begin{array}{ccc} \ell_\infty^n & \xrightarrow{v'} & \ell_1^m \\ d_\delta \downarrow & & \uparrow d_\gamma \\ \ell_p^n & \xrightarrow{c} & \ell_q^m \end{array}$$

where $\delta = (\delta_1, \dots, \delta_n)$, $\gamma = (\gamma_1, \dots, \gamma_m)$, d_δ and d_γ are the operators of multiplication and

$$\|d_\delta\| = \|\delta\|_{\ell_p} = 1, \|d_\gamma\| = \|\gamma\|_{\ell_{q'}} = 1 \text{ and } \|c\| < 1.$$

This factorization means that

$$v' = \sum_{i=1}^m \sum_{j=1}^n \gamma_i c(i, j) \delta_j e_j \otimes e_i.$$

Therefore, we have

$$\begin{aligned} v &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i c(i, j) \delta_j \varphi_{A,1}^{-1}(e_j) \otimes \psi_{B,1}^{-1}(e_i) \\ &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i, j)}{\mu_1(A_j) \mu_2(B_i)} \delta_j \chi_{A_j} \otimes \chi_{B_i}. \end{aligned}$$

We compute

$$\begin{aligned} u_\phi(v) &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i, j)}{\mu_1(A_j) \mu_2(B_i)} \delta_j \langle u_\phi(\chi_{A_j}), \chi_{B_i} \rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{c(i, j)}{\mu_1(A_j) \mu_2(B_i)} \delta_j \langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \rangle \end{aligned}$$

Define

$$\tilde{c} = \sum_{i=1}^m \sum_{j=1}^n \tilde{c}(i, j) \chi_{A_j} \otimes \chi_{B_i} \in L^{p'}(\Omega_1) \otimes L^q(\Omega_2),$$

where $\tilde{c}(i, j) = c_{i,j} \mu_1(A_j)^{-1/p'} \mu_2(B_i)^{-1/q}$.

Using the identification (7), it is easy to check that we have

$$\tilde{c} = \psi_{B,q}^{-1} \circ c \circ \varphi_{A,p} : S_{A,p} \mapsto L^q(\Omega_2).$$

Therefore,

$$\|\tilde{c}\|_v = \|c\|.$$

We have

$$\begin{aligned} u_\phi(v) &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \frac{\tilde{c}(i, j) \mu_1(A_j)^{1/p'} \mu_2(B_i)^{1/q}}{\mu_1(A_j) \mu_2(B_i)} \delta_j \langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \gamma_i \tilde{c}(i, j) \mu_1(A_i)^{-j1/p} \mu_2(B_i)^{-1/q'} \delta_j \langle T_\phi(\chi_{A_j} \otimes \chi_{B_i})(\chi_{A_j}), \chi_{B_i} \rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \left\langle T_\phi(\tilde{c}(i, j) \chi_{A_j} \otimes \chi_{B_i}) \left(\frac{\delta_j}{\mu_1(A_j)^{1/p}} \chi_{A_j} \right), \frac{\gamma_i}{\mu_2(B_i)^{1/q'}} \chi_{B_i} \right\rangle \\ &= \langle T_\phi(\tilde{c})(f), g \rangle_{L^q(\Omega_2), L^{q'}(\Omega_2)}, \end{aligned}$$

where

$$f = \sum_j \frac{\delta_j}{\mu_1(A_j)^{1/p}} \chi_{A_j} \quad \text{and} \quad g = \sum_i \frac{\gamma_i}{\mu_2(B_i)^{1/q'}} \chi_{B_i}.$$

Since $\|T_\phi\| \leq 1$, we deduce that

$$|u_\phi(v)| \leq \|T_\phi(\tilde{c})\| \|f\|_p \|g\|_{q'} \leq \|\tilde{c}\| \|\delta\|_{\ell_p} \|\gamma\|_{\ell_{q'}} = \|c\| \leq 1.$$

Conversely, assume that $u_\phi \in \mathcal{L}_{q,p'}(L^1(\Omega_1), L^\infty(\Omega_2))$ with $L_{q,p'}(u_\phi) \leq 1$. To prove that ϕ is a Schur multiplier, we will use Proposition 2.3. Let $n, m \in \mathbb{N}^*$, $A = (A_1, \dots, A_n) \in \mathcal{A}_{n,\Omega_1}$ and $B = (B_1, \dots, B_m) \in \mathcal{A}_{m,\Omega_2}$. Set

$$\phi_{ij} = \frac{1}{\mu_1(A_j)\mu_2(B_i)} \int_{A_j \times B_i} \phi \, d\mu_1 d\mu_2.$$

We want to show that the Schur multiplier on $\mathcal{B}(\ell_p^n, \ell_q^m)$ associated to the family $m = (\phi_{ij})_{i,j}$ has a norm less than 1. To prove that, let $c = \sum_{i,j} c(i,j) e_j \otimes e_i \in \mathcal{B}(\ell_p^n, \ell_q^m)$, $x = (x_j)_{j=1}^n$, $y = (y_i)_{i=1}^m$ in \mathbb{C} be such that $\|c\| \leq 1$, $\|x\|_{\ell_p^n} = 1$, $\|y\|_{\ell_{q'}^m} = 1$. We have to show that

$$|\langle [T_m(c)](x), y \rangle_{\ell_q^m, \ell_{q'}^m}| \leq 1.$$

This inequality can be rewritten as

$$(19) \quad \left| \sum_{i,j} c(i,j) \frac{x_j y_i}{\mu_1(A_j)\mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) \right| \leq 1.$$

Let $v = \sum_{i,j} x_j c(i,j) y_i e_j \otimes e_i$. According to (18), $\alpha'_{q,p'}(v) \leq 1$. Now, let $\tilde{v} = \sum_{i,j} x_j c(i,j) y_i \varphi_{A,1}^{-1}(e_j) \otimes \psi_{B,1}^{-1}(e_i)$. We have

$$\alpha'_{q,p'}(\tilde{v}) = \alpha'_{q,p'}(v) \leq 1$$

and

$$\tilde{v} = \sum_{i,j} \frac{x_j c(i,j) y_i}{\mu_1(A_j)\mu_2(B_i)} \chi_{A_j} \otimes \chi_{B_i}.$$

By assumption, $L_{q,p'}(u_\phi) \leq 1$, which implies that

$$\begin{aligned} |\langle u_\phi, \tilde{v} \rangle| &= \left| \sum_{i,j} c(i,j) \frac{x_j y_i}{\mu_1(A_j)\mu_2(B_i)} \left(\int_{A_j \times B_i} \phi \right) \right| \\ &\leq \alpha'_{q,p'}(\tilde{v}) \\ &\leq 1, \end{aligned}$$

and this is precisely the inequality (19). □

Theorem 3.2 and Remark 3.4 allow us to reformulate the previous theorem. The following two corollaries are generalizations of Theorem 1.1.

Corollary 4.3. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two localizable measure spaces and let $\phi \in L^\infty(\Omega_1 \times \Omega_2)$. Let $1 \leq q \leq p \leq \infty$. The following statements are equivalent :*

- (i) ϕ is a Schur multiplier on $\mathcal{B}(L^p(\Omega_1), L^q(\Omega_2))$.
- (ii) There are a measure space (a probability space when $p \neq q$) (Ω, μ) , operators $R \in \mathcal{B}(L^1(\Omega_1), L^p(\mu))$ and $S \in \mathcal{B}(L^q(\mu), L^\infty(\Omega_2))$ such that $u_\phi = S \circ I \circ R$

$$\begin{array}{ccc}
L^1(\Omega_1) & \xrightarrow{u_\phi} & L^\infty(\Omega_2) \\
R \downarrow & & \uparrow S \\
L^p(\mu) & \xrightarrow{I} & L^q(\mu)
\end{array}$$

where I is the inclusion mapping.

In the following cases, (i) and (ii) are equivalent to :

If $1 < q \leq p < +\infty$:

(iii) There are a measure space (a probability space when $p \neq q$) (Ω, μ) , $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that, for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \langle a(s), b(t) \rangle.$$

If $1 = q < p < +\infty$:

(iii) There are a probability space (Ω, μ) , $a \in L^\infty(\mu_1 \times \mu)$ and $b \in L^\infty(\mu_2, L^{q'}(\mu))$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \langle a(s, \cdot), b(t) \rangle.$$

If $1 < q < +\infty$ and $p = +\infty$:

(iii) There are a probability space (Ω, μ) , $a \in L^\infty(\mu_1, L^p(\mu))$ and $b \in L^\infty(\mu_2 \times \mu)$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \langle a(s), b(t, \cdot) \rangle.$$

If $q = 1$ and $p = +\infty$:

(iii) There are a probability space (Ω, μ) , $a \in L^\infty(\mu_1 \times \mu)$ and $b \in L^\infty(\mu_2 \times \mu)$ such that for almost every $(s, t) \in \Omega_1 \times \Omega_2$,

$$\phi(s, t) = \langle a(s, \cdot), b(t, \cdot) \rangle.$$

In this case, $\|T_\phi\| = \inf \|R\| \|I\| \|S\| = \inf \|a\| \|b\|$.

Remark 4.4. In the previous corollary, the condition (ii) implies that every $\phi \in L^\infty(\Omega_1 \times \Omega_2)$ is a Schur multiplier on $\mathcal{B}(L^1(\Omega_1), L^1(\Omega_2))$ and on $\mathcal{B}(L^\infty(\Omega_1), L^\infty(\Omega_2))$.

In the discrete case, the previous corollary can be reformulated as follow.

Corollary 4.5. Let $\phi = (c_{ij})_{i,j \in \mathbb{N}} \subset \mathbb{C}$, $C \geq 0$ be a constant and let $1 \leq q \leq p \leq +\infty$. The following are equivalent :

- (i) ϕ is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$ with norm $\leq C$.
- (ii) There exist a measure space (a probability space when $p \neq q$) (Ω, μ) and two bounded sequences $(x_j)_j$ in $L^p(\mu)$ and $(y_i)_i$ in $L^{q'}(\mu)$ such that

$$\forall i, j \in \mathbb{N}, c_{ij} = \langle x_j, y_i \rangle \text{ and } \sup_i \|y_i\|_{q'} \sup_j \|x_j\|_p \leq C.$$

4.2. An application : the main triangle projection. Let $m_{ij} = 1$ if $i \leq j$ and $m_{ij} = 0$ otherwise. Let T_m be the Schur multiplier associated with the family $m = (m_{ij})$. For any infinite matrix $A = [a_{ij}]$, $T_m(A)$ is the matrix $[b_{ij}]$ with $b_{ij} = a_{ij}$ if $i \leq j$ and $b_{ij} = 0$ otherwise. For that reason, T_m is called the main triangle projection. Similary, we define the n -th main triangle projection as the Schur multiplier on $\mathcal{M}_n(\mathbb{C})$ associated with the family $m_n = (m_{ij}^n)_{1 \leq i, j \leq n}$ where $m_{ij}^n = 1$ if $i \leq j$ and $m_{ij}^n = 0$ otherwise. In [8], Kwapień and Pelczyński proved that if $1 \leq q \leq p \leq +\infty, p \neq 1, q \neq +\infty$, there exists a constant $K > 0$ such that for all n ,

$$\|T_{m_n} : \mathcal{B}(\ell_p^n, \ell_q^n) \rightarrow \mathcal{B}(\ell_p^n, \ell_q^n)\| \geq K \ln(n),$$

and this order of growth is obtained for the Hilbert matrices. Those estimates imply that T_m is not bounded on $\mathcal{B}(\ell_p, \ell_q)$. Bennett proved in [2] that when $1 < p < q < \infty$, T_m is bounded from $\mathcal{B}(\ell_p, \ell_q)$ into itself.

The results obtained in subsection 4.1 allow us to give a very short proof of the unbounded case.

Proposition 4.6. *Let $1 \leq q \leq p \leq +\infty, p \neq 1, q \neq +\infty$. Then T_m is not bounded on $\mathcal{B}(\ell_p, \ell_q)$.*

Proof. Assume that T_m is bounded on $\mathcal{B}(\ell_p, \ell_q)$. By Corollary 4.3, there exist a measure space (Ω, μ) , $(a_n)_n \in L^p(\mu)$ and $(b_n)_n \in L^{q'}(\mu)$ two bounded sequences such that, for all $i, j \in \mathbb{N}$,

$$(20) \quad m_{ij} = \langle a_j, b_i \rangle.$$

By boundedness, $(a_n)_n$ and $(b_n)_n$ admit an accumulation point $a \in L^p(\mu)$ and $b \in L^{q'}(\mu)$ respectively for the weak-* topology. Fix $i \in \mathbb{N}$. For all $j \geq i$, we have

$$\langle a_i, b_j \rangle = 1$$

so that we get

$$\langle a_i, b \rangle = 1.$$

This equality holds for any i hence

$$\langle a, b \rangle = 1.$$

Now fix $j \in \mathbb{N}$. For all $i > j$ we have

$$\langle a_i, b_j \rangle = 0.$$

From this, we deduce as above that

$$\langle a, b \rangle = 0.$$

We obtained a contradiction so T_m cannot be bounded. □

As a consequence, we have, by Proposition 2.3 :

Corollary 4.7. *Let $1 \leq q \leq p \leq +\infty, p \neq 1, q \neq +\infty$. Let $\Omega_1 = \Omega_2 = \mathbb{R}$ with the Lebesgue measure. Then $\phi \in L^\infty(\mathbb{R}^2)$ defined by*

$$\phi(s, t) := \begin{cases} 1, & \text{if } s + t \geq 0 \\ 0 & \text{if } s + t < 0 \end{cases}, \quad s, t \in \mathbb{R}$$

is not a Schur multiplier on $\mathcal{B}(L^p(\mathbb{R}), L^q(\mathbb{R}))$.

Remark 4.8. One could wonder whether the results of subsection 4.1 can be extended to the case $1 \leq p < q \leq +\infty$, that is, if the boundedness of T_ϕ on $\mathcal{B}(L^p, L^q)$ implies that u_ϕ has a certain factorization. The fact that if $p < q$ the main triangle projection is bounded tells us that m is a Schur multiplier on $\mathcal{B}(\ell_p, \ell_q)$. Nevertheless, the argument used in the previous proof shows that m cannot have a factorization like in (20). Therefore, the case $p < q$ is more tricky. For the discrete case, one can find in [3, Theorem 4.3] a necessary and sufficient condition for a family $(m_{i,j}) \subset \mathbb{C}$ to be a Schur multiplier, for all values of p and q , using the theory of q -absolutely summing operators.

5. INCLUSION THEOREMS

In this section, we denote by $\mathcal{M}(p, q)$ the space of Schur multipliers on $\mathcal{B}(\ell_p, \ell_q)$.

First, we recall the inclusion relationships between the spaces $\mathcal{M}(p, q)$. Then we will establish new results as applications of those obtained in Section 4.1.

Theorem 5.1. [3, Theorem 6.1] *Let $p_1 \geq p_2$ and $q_1 \leq q_2$ be given. Then $\mathcal{M}(p_1, q_1) \subset \mathcal{M}(p_2, q_2)$ with equality in the following cases:*

- (i) $p_1 = p_2 = 1$,
- (ii) $q_1 = q_2 = \infty$,
- (iii) $q_2 \leq 2 \leq p_2$,
- (iv) $q_2 < p_1 = p_2 < 2$,
- (v) $2 < q_1 = q_2 < p_2$.

Let (Ω_1, μ_1) and (Ω_2, μ_2) be two measure spaces. If $\mathcal{M}(p_1, q_1) \subset \mathcal{M}(p_2, q_2)$, then using Proposition 2.3 we have that any Schur multiplier on $\mathcal{B}(L^{p_1}(\Omega_1), L^{q_1}(\Omega_2))$ is a Schur multiplier on $\mathcal{B}(L^{p_2}(\Omega_1), L^{q_2}(\Omega_2))$. Hence, the results in the previous theorem hold true for all the Schur multipliers on $\mathcal{B}(L^p, L^q)$.

In the sequel, we will need the notion of type for a Banach space X , for which we refer e.g. to [1]. Let $(\mathcal{E}_i)_{i \in \mathbb{N}}$ be a sequence of independent Rademacher random variables. We have the following definition.

Definition 5.2. *A Banach space X is said to have Rademacher type p (in short, type p) for some $1 \leq p \leq 2$ if there is a constant C such that for every finite set of vectors $(x_i)_{i=1}^n$ in X ,*

$$(21) \quad \left(\mathbb{E} \left\| \sum_{i=1}^n \mathcal{E}_i x_i \right\|^p \right)^{1/p} \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$

The smallest constant C for which (21) holds is called the type- p constant of X .

We will use the fact that for $1 \leq p \leq 2$, L^p -spaces have type p and if $2 < p < +\infty$, L^p -spaces have type 2 and that those are the best types for infinite dimensional L^p -spaces (see for instance [1, Theorem 6.2.14]). We will also use the fact that the type is stable by passing to quotients. Namely, if X has type p and $E \subset X$ is a closed subspace, then X/E has type p .

Proposition 5.3. (i) If $1 \leq q < p \leq 2$, then

$$\mathcal{M}(q, 1) \not\subseteq \mathcal{M}(p, p).$$

Consequently, for any $1 \leq r \leq q$,

$$\mathcal{M}(q, r) \not\subseteq \mathcal{M}(p, p).$$

(ii) If $2 \leq p < q \leq r$, then

$$\mathcal{M}(r, q) \not\subseteq \mathcal{M}(p, p).$$

(iii) If $1 < q < 2 < p < +\infty$ or $1 < p < 2 < q < +\infty$, then

$$\mathcal{M}(q, q) \not\subseteq \mathcal{M}(p, p).$$

To prove this proposition, we will need the following definitions and lemma.

Definition 5.4. Let X and Y be Banach spaces. A map $s : X \rightarrow Y$ is a quotient map if s is surjective and for all $y \in Y$ with $\|y\| < 1$, there exists $x \in X$ such that $\|x\| < 1$ and $s(x) = y$. This is equivalent to the fact that the injective map $\hat{s} : X/\ker(s) \rightarrow Y$ induced by s is a surjective isometry.

Definition 5.5. Let X and Y be Banach spaces, $u \in \mathcal{B}(X, Y)$ and $1 \leq p \leq \infty$. We say that $u \in SQ_p(X, Y)$ if there exists a closed subspace Z of a quotient of a L^p -space and two operators $A \in \mathcal{B}(X, Z)$ and $B \in \mathcal{B}(Z, Y)$ such that $u = BA$.

Then $\|u\|_{SQ_p} = \inf \|A\| \|B\|$ defines a norm on $SQ_p(X, Y)$ and $(SQ_p(X, Y), \|\cdot\|_{SQ_p})$ is a Banach space.

Lemma 5.6. Let W, X, Y, Z be Banach spaces and let $u \in \mathcal{B}(X, Y)$, $s \in \mathcal{B}(W, X)$, $v \in \mathcal{B}(Y, Z)$ such that s is a quotient map, v is a linear isometry and $vus \in \Gamma_p(W, Z)$. Then $u \in SQ_p(X, Y)$.

Proof. By assumption, there exist a L^p -space U and two operators $a \in \mathcal{B}(W, U)$ and $b \in \mathcal{B}(U, Z)$ such that the following diagram commutes

$$\begin{array}{ccccccc} W & \xrightarrow{s} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\ & \searrow a & & & & & \nearrow b \\ & & U & & & & \end{array}$$

Since v is an isometry, $V := v(Y) \subset Z$ is isometrically isomorphic to Y . Let $\psi : Y \rightarrow V$ be the isometric isomorphism induced by v .

Set $F := \{x \in U \text{ such that } b(x) \in V\}$. Since $vus = ba$, we have, for all $w \in W$, $v(us(w)) = b(a(w))$, so that $a(w) \in F$. This implies that $a(W) \subset F$. We still denote by a the mapping $a : W \rightarrow F$ and by b the restriction of b to F . Denote by \hat{b} the mapping $\hat{b} = \psi^{-1} \circ b : F \rightarrow Y$. Then we have the following commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{s} & X & \xrightarrow{u} & Y \\ & \searrow a & & & \nearrow \hat{b} \\ & & F & & \end{array}$$

Now, set $E := \overline{a(\ker(s))}$ and let $Q : F \rightarrow F/E$ be the canonical mapping. Clearly, $Q \circ a : W \rightarrow F/E$ vanishes on $\ker(s)$, so that we have a mapping

$$\widehat{Q \circ a} : W/\ker(s) \rightarrow F/E$$

induced by $Q \circ a$.

Since s is a quotient map, we denote by \hat{s} the isometric isomorphism

$$\hat{s} : W/\ker(s) \rightarrow X.$$

Define

$$A = \widehat{Q \circ a} \circ \hat{s}^{-1} : X \rightarrow F/E.$$

\hat{b} vanishes on E so that we have a mapping

$$B : F/E \rightarrow Y.$$

Finally, it is easy to check that $u = BA$, that is, we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & \searrow A & \nearrow B \\ & F/E & \end{array}$$

which concludes the proof. \square

Remark 5.7. To prove Lemma 5.6, one can use a result of Kwapien characterizing elements of SQ_p , as follows : a Banach space X is isomorphic to an SQ_q -space if and only if there exists a constant $K \geq 1$ such that for any $n \geq 1$, for any $n \times n$ matrix $[a_{ij}]$ and for any x_1, \dots, x_n in X ,

$$\left(\sum_i \left\| \sum_j a_{ij} x_j \right\|^q \right)^{1/q} \leq K \| [a_{ij}] : \ell_q^n \rightarrow \ell_q^n \| \left(\sum_j \|x_j\|^q \right)^{1/q}.$$

However, the proof presented in this paper also works if we replace in the statement of the lemma Γ_p (respectively SQ_p) by the space of operators that can be factorized by some Banach space L (respectively by a subspace of a quotient of L).

Proof of Proposition 5.3. (i). Let $\Omega := [0, 1]$ and λ be the Lebesgue measure on Ω . Let $I_q : L^q(\lambda) \rightarrow L^1(\lambda)$ be the inclusion mapping. By the classical Banach space theory (see [1, Theorem 2.3.1] and [1, Theorem 2.5.7]) there exist a quotient map $\sigma : \ell_1 \twoheadrightarrow L^q(\lambda)$ and an isometry $J : L^1(\lambda) \hookrightarrow \ell_\infty$. Let $\phi \in \ell_\infty(\mathbb{N}^2)$ be such that

$$u_\phi = J I_q \sigma$$

(by (6) any continuous linear map $\ell_1 \rightarrow \ell_\infty$ is a certain u_ϕ for $\phi \in L^\infty(\mathbb{N} \times \mathbb{N})$). We have the following factorization

$$\begin{array}{ccc} \ell_1 & \xrightarrow{u_\phi} & \ell_\infty \\ \sigma \downarrow & & \uparrow J \\ L^q(\lambda) & \xrightarrow{I_q} & L^1(\lambda) \end{array}$$

According to Theorem 4.3, $\phi \in \mathcal{M}(q, 1)$.

Assume that $\phi \in \mathcal{M}(p, p)$. Then, again by Theorem 4.3, we have $u_\phi \in \Gamma_p(\ell_1, \ell_\infty)$ and therefore, by Lemma 5.6, there exist an SQ_p -space X and two operators $\alpha \in \mathcal{B}(L^q(\lambda), X)$ and $\beta \in \mathcal{B}(X, L^1(\lambda))$ such that $I_q = \beta\alpha$.

Let $(\mathcal{E}_i)_{i \in \mathbb{N}}$ be a sequence of independant Rademacher random variables. Let $n \in \mathbb{N}^*$ and $f_1, \dots, f_n \in L^q(\lambda)$.

$$\mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j f_j \right\|_{L^1(\lambda)} = \mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j \beta \alpha(f_j) \right\|_{L^1(\lambda)} \leq \|\beta\| \mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j \alpha(f_j) \right\|_X.$$

But X has type p so there exists a constant $C_1 > 0$ such that

$$\mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j f_j \right\|_{L^1(\lambda)} \leq C_1 \|\beta\| \left(\sum_{j=1}^n \|\alpha(f_j)\|_X^p \right)^{1/p} \leq C_1 \|\beta\| \|\alpha\| \left(\sum_{j=1}^n \|f_j\|_{L^q(\lambda)}^p \right)^{1/p}.$$

By Khintchine inequality, there exists $C_2 > 0$ such that

$$\left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\|_{L^1(\lambda)} \leq C_2 \mathbb{E} \left\| \sum_{j=1}^n \mathcal{E}_j f_j \right\|_{L^1(\lambda)}.$$

Thus, setting $K := C_1 C_2 \|\alpha\| \|\beta\|$, we obtained the inequality

$$\left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{1/2} \right\|_{L^1(\lambda)} \leq K \left(\sum_{j=1}^n \|f_j\|_{L^q(\lambda)}^p \right)^{1/p}.$$

Let E_1, \dots, E_n be disjoint measurable subsets of $[0, 1]$ such that for all $1 \leq j \leq n$, $\lambda(E_j) = \frac{1}{n}$. Set $f_j := \chi_{E_j}$. Then

$$\sum_j |f_j|^2 = 1 \quad \text{and} \quad \|f\|_{L^q(\lambda)} = n^{-1/q}.$$

Hence, applying the previous inequality to the f_j 's, we obtain

$$1 \leq K n^{1/p-1/q}.$$

Since $q < p$, this inequality can't hold for all n , so we obtained a contradiction.

Finally, notice that if $1 \leq r \leq q$, then by Theorem 5.1, $\mathcal{M}(q, 1) \subset \mathcal{M}(q, r)$. Thus, $\mathcal{M}(q, r) \not\subset \mathcal{M}(p, p)$.

(ii). By Proposition 2.3 and using duality, it is easy to prove that for all $s, t \in [1, \infty]$, ϕ is a Schur multiplier on $\mathcal{B}(\ell_s, \ell_t)$ if and only if $\tilde{\phi}$ is a Schur multiplier on $\mathcal{B}(\ell_{t'}, \ell_{s'})$, where $\tilde{\phi}$ is defined for all $i, j \in \mathbb{N}$ by $\tilde{\phi}(i, j) = \phi(j, i)$.

Let $2 \leq p < q \leq r$. Then $1 \leq r' \leq q' < p' \leq 2$. If we assume that $\mathcal{M}(r, q) \subset \mathcal{M}(p, p)$ then the latter implies $\mathcal{M}(q', r') \subset \mathcal{M}(p', p')$, which is, by (i), a contradiction. This proves (ii).

(iii). By duality, it is enough to consider the case $1 < q < 2 < p < +\infty$. Assume that $\mathcal{M}(q, q) \subset \mathcal{M}(p, p)$. Using the notations introduced in the proof of (i), let $\sigma : \ell_1 \rightarrow \ell_q$ be a quotient map and $J : \ell_q \rightarrow \ell_\infty$ be an isometry. Let $\phi \in L^\infty(\mathbb{N} \times \mathbb{N})$ be such that

$$u_\phi = JI_{\ell_q}\sigma,$$

where $I_{\ell_q} : \ell_q \rightarrow \ell_q$ is the identity map. Then $\phi \in \mathcal{M}(q, q)$. By assumption, $\phi \in \mathcal{M}(p, p)$. By Lemma 5.6, this implies that $I_{\ell_q} \in SQ_p(\ell_q, \ell_q)$. Clearly, this implies that ℓ_q is isomorphic to an SQ_p -space. But ℓ_q does not have type 2 and any SQ_p has type 2. This is a contradiction, so $\mathcal{M}(q, q) \not\subset \mathcal{M}(p, p)$. \square

Theorem 5.8. *We have $\mathcal{M}(q, q) \subset \mathcal{M}(p, p)$ if and only if $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p \leq +\infty$.*

Proof. By Proposition 5.3 and duality, we only have to show that when $1 \leq p \leq q \leq 2$, $\mathcal{M}(q, q) \subset \mathcal{M}(p, p)$.

We saw in the proof Proposition of 5.3 (iii) that if $\mathcal{M}(q, q) \subset \mathcal{M}(p, p)$ then ℓ_q is isomorphic to an SQ_p -space. The converse holds true. Indeed, assume that ℓ_q is isomorphic to an SQ_p -space. Then by approximation, any L^q -space is isomorphic to an SQ_p -space. Hence any element of $\Gamma_q(\ell_1, \ell_\infty)$ factors through an SQ_p -space. By the lifting property of ℓ_1 and the extension property of ℓ_∞ , this implies that any element of $\Gamma_q(\ell_1, \ell_\infty)$ factors through an L^p -space, that is $\Gamma_q(\ell_1, \ell_\infty) \subset \Gamma_p(\ell_1, \ell_\infty)$. By Corollary 4.5, this implies that $\mathcal{M}(q, q) \subset \mathcal{M}(p, p)$.

Assume that $1 \leq p \leq q \leq 2$. By [1, Theorem 6.4.19], there exists an isometry from ℓ_q into an L^p -space, obtained by using q -stable processes. Hence, ℓ_q is an SQ_p -space. This concludes the proof. \square

Problem 5.9. Compare the other spaces of Schur multipliers. For example, if $1 < p \leq 2$, do we have

$$\mathcal{M}(p, 1) = \mathcal{M}(p, p)?$$

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